On a classical limit of q-deformed Whittaker functions

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Abstract. We provide a derivation of the Givental integral representation of the classical $\mathfrak{gl}_{\ell+1}$-Whittaker function as a limit $q \to 1$ of the q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function represented as a sum over the Gelfand-Zetlin patterns.

Introduction

The q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker functions can be defined as eigenfunctions of the q-deformed $\mathfrak{gl}_{\ell+1}$-Toda chains [Ru], [Et]. Among various eigenfunctions there exists a special class of eigenfunctions with the support in the positive Weyl chamber. By analogy with the classical case we call such functions the class one q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker functions. In [GLO1] an explicit representation of the class one q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function as a sum over the Gelfand-Zetlin patterns was proposed. This representation has remarkable integrality and positivity properties. Precisely each term in the sum is a positive integer multiplied by a weight factor q^{wt} and a character of the torus $U_1^{\ell+1}$. This allows to represent the q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function as a character of a $\mathbb{C}^* \times U_{\ell+1}$-module (i.e. it allows a categorification). The interpretation of the q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function as a character shall be considered as a q-version of the Shintani-Casselman-Shalika formula [Sh], [CS]. Indeed in the limit $q \to 0$ the q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function can be identified with the non-Archimedean Whittaker function and the representation of the q-deformed Whittaker function as a character reduces to the standard Shintani-Casselman-Shalika formula for non-Archimedean Whittaker function [Sh], [CS].

In the limit $q \to 1$ the q-Whittaker functions reproduces the classical Whittaker functions. It was pointed out in [GLO1] that in this limit an explicit sum type representation of the class one q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function turns into the Givental integral representation for the class one $\mathfrak{gl}_{\ell+1}$-Whittaker function [Gi] (see also [GKLO]). Thus the Givental integral representation shall be considered as the Archimedean counterpart of the Shintani-Casselman-Shalika formula (for more details on this interpretation see [GLO2], [GLO3], [G]). In this note we provide a precise description of the $q \to 1$ limit reducing the q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function to its classical analog and explicitly demonstrate that the Givental integral representation arises as a limit of the sum representation of the q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function. This result is given by Theorem 3.1. The established relation between a sum over the Gelfand-Zetlin patterns for $\mathfrak{gl}_{\ell+1}$ and the Givental integrals for $\mathfrak{gl}_{\ell+1}$ is a special case of a general relation between the Gelfand-Zetlin patterns and the Givental type integrals for classical series of Lie algebras [GLO4]. This relation elucidates the identification of the Givental and the Gelfand-Zetlin graphs noticed in [GLO4]. The relation between the Gelfand-Zetlin and Givental constructions described in this note should be also compared with the duality type relation introduced in [GLO5]. We are going to discuss the

general form of the relation between the Gelfand-Zetlin and the Givental constructions for classical Lie algebras elsewhere.

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1 q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function

In this Section we recall the explicit construction of the class one q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker functions derived in [GLO1]. Quantum q-deformed $\mathfrak{gl}_{\ell+1}$-Toda chain (see e.g. [Ru], [Et]) is defined by a set of $\ell+1$ mutually commuting functionally independent quantum Hamiltonians $\mathcal{H}_r^{\mathfrak{gl}_{\ell+1}}$, $r=1,\ldots,\ell+1$:

$$\mathcal{H}_r^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1}) \,=\, \sum_{I_r}\big(\widetilde{X}_{i_1}^{1-\delta_{i_2-i_1,1}}\cdot\ldots\cdot\widetilde{X}_{i_{r-1}}^{1-\delta_{i_r-i_{r-1},1}}\cdot\widetilde{X}_{i_r}^{1-\delta_{i_{r+1}-i_r,1}}\big)T_{i_1}\cdot\ldots\cdot T_{i_r}\,, \tag{1.1}$$

where $r=1,\ldots,\ell+1$ and $i_{r+1}=\ell+2$. The summation in (1.1) goes over all ordered subsets $I_r=\{i_1<i_2<\cdots<i_r\}$ of $\{1,2,\cdots,\ell+1\}$. Here we use the notations

$$T_if(\underline{p}_{\ell+1})=f(\underline{\widetilde{p}}_{\ell+1}),\qquad \widetilde{p}_{\ell+1,k}=p_{\ell+1,k}+\delta_{k,i},$$
$$\widetilde{X}_i=1-q^{p_{\ell+1,i}-p_{\ell+1,i+1}+1},\qquad i=1,\ldots,\ell,\qquad \widetilde{X}_{\ell+1}=1.$$

The corresponding eigenvalue problem can be written in the following form:

$$\mathcal{H}_r^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1})\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\ldots,z_{\ell+1}}(\underline{p}_{\ell+1}) \,=\, \Big(\sum_{I_r}\prod_{i\in I_r}z_i\Big)\ \Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\ldots,z_{\ell+1}}(\underline{p}_{\ell+1}), \tag{1.2}$$

and the first nontrivial Hamiltonian is given by

$$\mathcal{H}_1^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1}) \,=\, \sum_{i=1}^{\ell}(1-q^{p_{\ell+1,i}-p_{\ell+1,i+1}+1})T_i\,+\,T_{\ell+1}. \tag{1.3}$$

One of the main results of [GLO1] now can be formulated as follows. Given $\underline{p}_{\ell+1}=(p_{\ell+1,1},\ldots,p_{\ell+1,\ell+1})$ let us denote by $\mathcal{P}^{(\ell+1)}(\underline{p}_{\ell+1})$ a set of collections of the integer parameters $p_{k,i}$, $k=1,\ldots,\ell$, $i=1,\ldots,k$ satisfying the Gelfand-Zetlin conditions $p_{k+1,i}\geq p_{k,i}\geq p_{k+1,i+1}$. Let $\mathcal{P}_{\ell+1,\ell}(\underline{p}_{\ell+1})$ be a set of $\underline{p}_\ell=(p_{\ell,1},\ldots,p_{\ell,\ell})$, $p_{\ell,i}\in\mathbb{Z}$, satisfying the conditions $p_{\ell+1,i}\geq p_{\ell,i}\geq p_{\ell+1,i+1}$.

Theorem 1.1 *A common solution of the eigenvalue problem (1.2) can be written in the following form. For $\underline{p}_{\ell+1}$ being in the dominant domain $p_{\ell+1,1}\geq\ldots\geq p_{\ell+1,\ell+1}$, the solution is given by*

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\ldots,z_{\ell+1}}(\underline{p}_{\ell+1}) \,=\, \sum_{p_{k,i}\in\mathcal{P}^{(\ell+1)}(\underline{p}_{\ell+1})}\ \prod_{k=1}^{\ell+1}z_k^{\sum_ip_{k,i}-\sum_ip_{k-1,i}}$$
$$\times\frac{\prod\limits_{k=2}^{\ell}\prod\limits_{i=1}^{k-1}(p_{k,i}-p_{k,i+1})_q!}{\prod\limits_{k=1}^{\ell}\prod\limits_{i=1}^{k}(p_{k+1,i}-p_{k,i})_q!\ (p_{k,i}-p_{k+1,i+1})_q!}, \tag{1.4}$$

where we use the notation $(n)_q! = (1-q)...(1-q^n)$. *When* $\underline{p}_{\ell+1}$ *is outside the dominant domain we set*

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\dots,z_{\ell+1}}(p_{\ell+1,1},\dots,p_{\ell+1,\ell+1}) \,=\, 0.$$

Example 1.1 *Let* $\mathfrak{g}=\mathfrak{gl}_2$, $p_{2,1}:=p_1\in\mathbb{Z}$, $p_{2,2}:=p_2\in\mathbb{Z}$ *and* $p_{1,1}:=p\in\mathbb{Z}$. *The function*

$$\Psi^{\mathfrak{gl}_2}_{z_1,z_2}(p_1,p_2) = \sum_{p_2\le p\le p_1}\frac{z_1^p z_2^{p_1+p_2-p}}{(p_1-p)_q!(p-p_2)_q!}, \qquad p_1\ge p_2\,,$$

$$\Psi^{\mathfrak{gl}_2}_{z_1,z_2}(p_1,p_2)=0, \qquad p_1<p_2\,,$$

is a common eigenfunction of mutually commuting Hamiltonians

$$\mathcal{H}^{\mathfrak{gl}_2}_1 \,=\, (1-q^{p_1-p_2+1})T_1+T_2, \qquad \mathcal{H}^{\mathfrak{gl}_2}_2=T_1T_2.$$

The formula (1.4) can be easily rewritten in the recursive form.

Corollary 1.1 *The following recursive relation holds*

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\dots,z_{\ell+1}}(\underline{p}_{\ell+1}) \;=\; \sum_{\underline{p}_\ell\in\mathcal{P}_{\ell+1,\ell}(\underline{p}_{\ell+1})} \Delta(\underline{p}_\ell)\; z_{\ell+1}^{\sum_i p_{\ell+1,i}-\sum_i p_{\ell,i}}\; Q_{\ell+1,\ell}(\underline{p}_{\ell+1},\underline{p}_\ell|q)\Psi^{\mathfrak{gl}_\ell}_{z_1,\dots,z_\ell}(\underline{p}_\ell), \tag{1.5}$$

where

$$\begin{gathered} Q_{\ell+1,\ell}(\underline{p}_{\ell+1},\underline{p}_\ell|q) \;=\; \frac{1}{\prod\limits_{i=1}^{\ell}(p_{\ell+1,i}-p_{\ell,i})_q!\;(p_{\ell,i}-p_{\ell+1,i+1})_q!}, \\ \Delta(\underline{p}_\ell)=\prod_{i=1}^{\ell-1}(p_{\ell,i}-p_{\ell,i+1})_q!\;\;. \end{gathered} \tag{1.6}$$

The following representations of the class one q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function are a consequence of the positivity and integrality of the coefficients of the q-series expansions of each term in the sum (1.4) (see [GLO1] for details).

Proposition 1.1 *(i). There exists a* $\mathbb{C}^*\times GL_{\ell+1}(\mathbb{C})$ *module* V *such that the common eigenfunction* (1.4) *of the q-deformed Toda chain allows the following representation for* $p_{\ell+1,1}\ge p_{\ell+1,2}\ge\ldots\ge p_{\ell+1,\ell+1}$*:*

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}(\underline{p}_{\ell+1}) \;=\; \mathrm{Tr}_{\,V}\, q^{L_0}\prod_{i=1}^{\ell+1}q^{\lambda_iH_i}, \tag{1.7}$$

where $H_i:=E_{i,i}$, $i=1,\dots,\ell+1$ *are Cartan generators of* $\mathfrak{gl}_{\ell+1}=\mathrm{Lie}(GL_{\ell+1})$ *and* L_0 *is a generator of* $\mathrm{Lie}(\mathbb{C}^*)$.

(ii). There exists a finite-dimensional $\mathbb{C}^*\times GL_{\ell+1}(\mathbb{C})$ *module* V_f *such that the following representation holds for* $p_{\ell+1,1}\ge p_{\ell+1,2}\ge\ldots\ge p_{\ell+1,\ell+1}$*:*

$$\widetilde{\Psi}^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}(\underline{p}_{\ell+1}) \;=\; \Delta(\underline{p}_{\ell+1})\;\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}(\underline{p}_{\ell+1})=\mathrm{Tr}_{\,V_f}\,q^{L_0}\prod_{i=1}^{\ell+1}q^{\lambda_i\,H_i}. \tag{1.8}$$

The module V *entering* (1.7) *and the module* V_f *entering* (1.8) *have a structure of modules under the action of (quantum) affine Lie algebras [GLO1].*

2 Classical limit of q-deformed Toda chain

In this Section we define a limit $q \to 1$ of the q-deformed $\mathfrak{gl}_{\ell+1}$-Toda chain reproducing the standard $\mathfrak{gl}_{\ell+1}$-Toda chain. We provide an explicit check that the first two generators of the ring of quantum Hamiltonians of $\mathfrak{gl}_{\ell+1}$-Toda chain arise as a limit of some combinations of the following quantum Hamiltonians of the q-deformed Toda chain

$$\mathcal{H}_1^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1}|q) \,=\, \sum_{i=1}^{\ell} \Big(1 \,-\, q^{p_{\ell+1,i}-p_{\ell+1,i+1}+1}\Big)\, T_i \,+\, T_{\ell+1},$$

$$\mathcal{H}_{\ell+1}^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1}|q) \,=\, T_1 T_2 \cdots T_{\ell+1}.$$

Let us introduce the following parametrization:

$$q = e^{-\epsilon}, \qquad p_{\ell+1,k} = (\ell+2-2k)m(\epsilon) + x_{\ell+1,k}\epsilon^{-1}. \tag{2.1}$$

Here $m(\epsilon) \in \mathbb{Z}$ is given by

$$m(\epsilon) \,=\, -[\epsilon^{-1} \ln \epsilon],$$

and $[x] \in \mathbb{Z}$ is the integer part of x.

Proposition 2.1 *The following limiting relations hold:*

$$H_1^{\mathfrak{gl}_{\ell+1}}(\underline{x}_{\ell+1}) \,=\, \lim_{\epsilon \to 0} \frac{1}{\epsilon} \Big[\mathcal{H}_1^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1}(x,\epsilon)|q(\epsilon)) - (\ell+1)\Big],$$

$$H_2^{\mathfrak{gl}_{\ell+1}}(\underline{x}_{\ell+1}) \,=\, -\lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \Big[\mathcal{H}_1^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1}(x,\epsilon)|q(\epsilon)) - \mathcal{H}_{\ell+1}^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1}(x,\epsilon)|q(\epsilon)) - \ell$$
$$+\frac{1}{2}(\mathcal{H}_{\ell+1}^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1}(x,\epsilon)|q(\epsilon)) - 1)^2\Big]$$

where $H_i^{\mathfrak{gl}_{\ell+1}}$, $i = 1, 2$ are the standard quantum Hamiltonians of the $\mathfrak{gl}_{\ell+1}$-Toda chain:

$$H_1^{\mathfrak{gl}_{\ell+1}}(\underline{x}_{\ell+1}) = \sum_{i=1}^{\ell+1} \frac{\partial}{\partial x_{\ell+1,i}},$$

$$H_2^{\mathfrak{gl}_{\ell+1}}(\underline{x}_{\ell+1}) \,=\, -\frac{1}{2}\sum_{i=1}^{\ell+1} \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{\ell} e^{x_{i+1}-x_i}.$$

Proof. Using the fact that $\exp(\epsilon[2(\epsilon)^{-1}\ln(\epsilon)] = \epsilon^2(1 + O(\epsilon^2/\ln\epsilon))$ we have

$$\mathcal{H}_1^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1}|q) \,=\, (\ell+1) \,+\, \epsilon \sum_{i=1}^{\ell+1} \frac{\partial}{\partial x_{\ell+1,i}} + \epsilon^2 \Big(\frac{1}{2}\sum_{i=1}^{\ell+1} \frac{\partial^2}{\partial x_{\ell+1,i}^2} \,-\, \sum_{k=1}^{\ell} e^{x_{\ell+1,k+1}-x_{\ell+1,k}}\Big) \,+\, O(\epsilon^3),$$

$$\mathcal{H}_{\ell+1}^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1}|q) \,=\, 1 \,+\, \epsilon \sum_{i=1}^{\ell+1} \frac{\partial}{\partial x_{\ell+1,i}} + \frac{1}{2}\epsilon^2 \sum_{i,j=1}^{\ell+1} \frac{\partial^2}{\partial x_{\ell+1,i}\partial x_{\ell+1,j}} \,+\, O(\epsilon^3).$$

Now the limiting formulas can be straightforwardly verified. We have

$$\mathcal{H}_1^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1}|q) - \mathcal{H}_{\ell+1}^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1}|q) - \ell = \epsilon^2\Big(-\frac{1}{2}\sum_{i\neq j}^{\ell+1} \frac{\partial^2}{\partial x_{\ell+1,i}\partial x_{\ell+1,j}} - \sum_{k=1}^{\ell} e^{x_{\ell+1,k+1}-x_{\ell+1,k}}\Big) + O(\epsilon^3),$$

$$\frac{1}{2}(\mathcal{H}_{\ell+1}^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1}|q) - 1)^2 = \frac{1}{2}\epsilon^2\Big(\sum_{i,j=1}^{\ell+1} \frac{\partial^2}{\partial x_{\ell+1,i}\partial x_{\ell+1,j}}\Big) + O(\epsilon^3),$$

and thus

$$\mathcal{H}_1^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1}|q) - \mathcal{H}_{\ell+1}^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1}|q) - \ell + \frac{1}{2}(\mathcal{H}_{\ell+1}^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1}|q) - 1)^2$$
$$= \epsilon^2\Big(\frac{1}{2}\sum_{i=1}^{\ell+1} \frac{\partial^2}{\partial x_{\ell+1,i}^2} - \sum_{k=1}^{\ell} e^{x_{\ell+1,k+1}-x_{\ell+1,k}}\Big) + O(\epsilon^3).$$

□

It is easy to see that the eigenfunction problem (1.2) is transformed into the standard eigenfunction problem if we use the following parametrization of the spectral variables $z_i = e^{\imath\,\epsilon\lambda_i}$, $i = 1,\dots,\ell+1$.

3 Classical limit of class one Whittaker function

In the limit $q \to 1$ defined in the previous Section the class one solution (1.4) of the q-deformed $\mathfrak{gl}_{\ell+1}$-Toda chain should goes to the class one solution of the classical $\mathfrak{gl}_{\ell+1}$-Toda chain. In the classical setting an integral representation for class one $\mathfrak{gl}_{\ell+1}$-Whittaker function was constructed by Givental [Gi], (see [GKLO] for a choice of the contour realizing class one condition)

$$\psi^{\mathfrak{gl}_{\ell+1}}_{\lambda_1,\dots,\lambda_{\ell+1}}(x_1,\dots,x_{\ell+1}) = \int_C \prod_{k=1}^{\ell} d\underline{x}_k\; e^{\mathcal{F}^{\mathfrak{gl}_{\ell+1}}(x)}, \tag{3.1}$$

and the function $\mathcal{F}^{\mathfrak{gl}_{\ell+1}}(x)$ is given by

$$\mathcal{F}^{\mathfrak{gl}_{\ell+1}}(x) = \imath\sum_{n=1}^{\ell+1}\lambda_n\Big(\sum_{i=1}^{n} x_{n,i} - \sum_{i=1}^{n-1} x_{n-1,i}\Big) - \sum_{k=1}^{\ell}\sum_{i=1}^{k}\Big(e^{x_{k,i}-x_{k+1,i}} + e^{x_{k+1,\,i+1}-x_{k,i}}\Big). \tag{3.2}$$

Here $C \subset N_+$ is a small deformation of the subspace $\mathbb{R}^{\frac{(\ell+1)\ell}{2}} \subset \mathbb{C}^{\frac{(\ell+1)\ell}{2}}$ making the integral (3.2) convergent. Besides, we use the following notation: $\underline{\lambda} = (\lambda_1,\dots,\lambda_{\ell+1})$; $x_i := x_{\ell+1,i}$, $i = 1,\dots,\ell+1$.

The integral representation (3.1) allows a recursive presentation analogous to (1.5)

$$\psi^{\mathfrak{gl}_{\ell+1}}_{\lambda_1,\dots,\lambda_{\ell+1}}(x_1,\dots,x_{\ell+1}) = \int\limits_{\mathbb{R}^\ell} d\underline{x}_\ell\; Q^{\mathfrak{gl}_{\ell+1}}_{\mathfrak{gl}_\ell}(\underline{x}_{\ell+1};\underline{x}_\ell;\lambda_{\ell+1})\psi^{\mathfrak{gl}_\ell}_{\lambda_1,\dots,\lambda_\ell}(x_{\ell,1},\dots,x_{\ell\ell}), \tag{3.3}$$

where

$$Q^{\mathfrak{gl}_{\ell+1}}_{\mathfrak{gl}_\ell}(\underline{x}_{\ell+1};\underline{x}_\ell;\lambda_{\ell+1}) = \exp\Big\{\imath\lambda_{\ell+1}\Big(\sum_{i=1}^{\ell+1} x_{\ell+1,i} - \sum_{i=1}^{\ell} x_{\ell,\,i}\Big) - \sum_{i=1}^{\ell}\Big(e^{x_{\ell,\,i}-x_{\ell+1,\,i}} + e^{x_{\ell+1,\,i+1}-x_{\ell,\,i}}\Big)\Big\}, \tag{3.4}$$

and we assume $Q^{\mathfrak{gl}_1}_{\mathfrak{gl}_0}(x_{11};\,\lambda_1) = e^{\imath\lambda_1 x_{11}}$.

In the following we demonstrate that in the previously defined limit $q \to 1$ the class one q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function given by the sum (1.4) indeed turns into the classical class one $\mathfrak{gl}_{\ell+1}$-Whittaker function given by the integral representation (3.1). In particular iterative formula (1.5) turns into (3.3). For this purpose we need the following asymptotic of the q-factorials entering (1.4).

Lemma 3.1 *Let us introduce the following functions*

$$f_\alpha(y,\epsilon) = (y/\epsilon + \alpha m(\epsilon))_q!, \qquad \alpha = 1, 2,$$

where $m(\epsilon) = -[\epsilon^{-1}\ln\epsilon]$, $q = e^{-\epsilon}$. *Then for* $\epsilon \to +0$ *the following expansions hold:*

$$f_1(y,\epsilon) = e^{A(\epsilon)+e^{-y}+O(\epsilon)}\,; \tag{3.5}$$

$$f_2(y,\epsilon) = e^{A(\epsilon)+O(\epsilon^{\alpha-1})}, \tag{3.6}$$

where $A(\epsilon) = -\frac{\pi^2}{6}\frac{1}{\epsilon} - \frac{1}{2}\ln\frac{\epsilon}{2\pi}$.

Proof. Taking into account the identity

$$\ln\prod_{n=1}^{N}(1-q^n) = \sum_{n=1}^{N}\ln(1-q^n) = -\sum_{n=1}^{N}\sum_{r=1}^{+\infty}\frac{1}{r}q^{nr} = -\sum_{r=1}^{+\infty}\frac{q^r}{r}\left(\frac{1-q^{Nr}}{1-q^r}\right),$$

and using the substitution $q = e^{-\epsilon}$, $N = \epsilon^{-1}y + \alpha m(\epsilon)$ we obtain

$$\ln\, f_\alpha(y,\epsilon) = -\sum_{r=1}^{+\infty}\frac{e^{-r\epsilon}}{r}\left(\frac{1-e^{-\alpha r\epsilon m(\epsilon)}e^{-ry}}{1-e^{-r\epsilon}}\right).$$

Now expanding the denominator over small ϵ we have

$$\ln\, f_\alpha(y,\epsilon) = -\sum_{r=1}^{+\infty}\frac{e^{-r\epsilon}}{r}\left(\frac{1-\epsilon^{\alpha r}e^{-ry}}{1-e^{-r\epsilon}}\right) + \cdots = -\sum_{r=1}^{+\infty}\frac{e^{-r\epsilon}}{r^2\epsilon}\left(\frac{1-\epsilon^{\alpha r}e^{-ry}}{1-\frac{1}{2}r\epsilon+\frac{1}{3!}r^2\epsilon^2+\cdots}\right) + \cdots,$$

and for the derivative we obtain

$$\partial_y \ln\, f_\alpha(y,\epsilon) \;=\; -\sum_{r=1}^{+\infty}\frac{1}{r\epsilon}\left(\frac{\epsilon^{\alpha r}e^{-ry-r\epsilon}}{1-\frac{1}{2}r\epsilon+\frac{1}{3!}r^2\epsilon^2+\cdots}\right) + \cdots = \sum_{k=-1}^{+\infty} c_k\, I_{\alpha,k}(y,\epsilon),$$

where

$$I_{\alpha,k}(y,\epsilon) \;=\; \sum_{r=1}^{+\infty}\epsilon^{k+\alpha r}r^k e^{-yr} \;=\; \epsilon^k\sum_{r=1}^{+\infty}t^r r^k, \qquad t = e^{-y}\epsilon^\alpha,$$

and $c_{-1} = -1$. Let us separately analyze the term $I_{\alpha,-1}$ and the other terms $I_{\alpha,k\geq 0}$. We have

$$I_{\alpha,k\geq 0}(y,\epsilon) \;=\; \epsilon^k\left(t\frac{\partial}{\partial t}\right)^k\frac{1}{1-t} \;=\; \epsilon^k\frac{\partial^k}{\partial y^k}\frac{1}{1-\epsilon^\alpha e^{-y}},$$

and thus

$$I_{\alpha,k\geq 0} \;=\; \epsilon^{k+\alpha}e^{-y} + \cdots, \qquad \alpha = 1, 2.$$

Now consider the case of $k=-1$

$$c_{-1}I_{\alpha,-1}(y,\epsilon)=-\frac{1}{\epsilon}\sum_{r=1}^{+\infty}\frac{t^r}{r}=-\frac{1}{\epsilon}\ln(1-t)=-\frac{1}{\epsilon}\ln(1-\epsilon^{\alpha}e^{-y}))=\epsilon^{\alpha-1}e^{-y}+\cdots\,,\qquad t=e^{-y}\epsilon^{\alpha}.$$

This gives (3.5), (3.6) with an unknown $A(\epsilon)$. To calculate $A(\epsilon)$ we take $e^{-y}=0$ and notice that the resulting function does not depend on α. Thus we should calculate the asymptotic of the following function:

$$\ln f_{\alpha}(y,\epsilon)|_{e^y=0}=-\sum_{r=1}^{+\infty}\frac{e^{-r\epsilon}}{r}\left(\frac{1-\epsilon^{r\alpha}e^{-ry}}{1-e^{-r\epsilon}}\right)\Big|_{e^y=0}=-\sum_{r=1}^{+\infty}\frac{1}{r}\left(\frac{e^{-r\epsilon}}{1-e^{-r\epsilon}}\right)=$$

$$=-\sum_{n=1}^{+\infty}\sum_{r=1}^{+\infty}\frac{1}{r}e^{-nr\epsilon}=\ln\prod_{n=1}^{+\infty}(1-e^{-n\epsilon}).$$

It can be easily done using the modular properties

$$\eta(-\tau^{-1})\;=\;\sqrt{-\imath\tau}\;\eta(\tau),$$

of the Dedekind eta function

$$\eta(\tau)\;=\;e^{\frac{\imath\pi\tau}{12}}\;\prod_{n=1}^{\infty}(1-e^{2\pi\imath n\tau}).$$

Namely, taking $\tau=\frac{\imath\epsilon}{2\pi}$ we have

$$f_{\alpha}(y;\,\epsilon)\Big|_{e^{-y}=0}\;=\;\sqrt{2\pi\epsilon^{-1}}\,e^{-\frac{\pi^2}{6}\epsilon^{-1}}\;\prod_{n=1}^{\infty}\left(1-e^{-\epsilon^{-1}(2\pi)^2n}\right).$$

This allows to infer the following result for the leading coefficients in the asymptotic expansion of $\ln f_{\alpha}(y,\epsilon)\Big|_{e^{-y}=0}$:

$$A(\epsilon)\;=\;-\frac{1}{2}\ln\frac{\epsilon}{2\pi}-\frac{\pi^2}{6}\epsilon^{-1}\,,\qquad\qquad \epsilon\longrightarrow+0\,. \tag{3.7}$$

This completes the proof of Lemma. □

Theorem 3.1 *Let us use the following parametrization*

$$q=e^{-\epsilon},\qquad p_{\ell+1,k}=(\ell+2-2k)m(\epsilon)+\epsilon^{-1}x_{\ell+1,k}\,,\qquad z_k=e^{\imath\,\epsilon\lambda_k}\,, \tag{3.8}$$

where $k=1,\dots,\ell+1$, $m(\epsilon)=-\big[\epsilon^{-1}\ln\epsilon\big]$. *The integral representation* (3.1) *of the classical* $\mathfrak{gl}_{\ell+1}$*-Whittaker function is given by the following limit of the* q*-deformed class one* $\mathfrak{gl}_{\ell+1}$*-Whittaker function represented as a sum* (1.4)

$$\psi^{\mathfrak{gl}_{\ell+1}}_{\lambda_1,\dots,\lambda_{\ell+1}}(x_1,\dots,x_{\ell+1})\;=\;\lim_{\epsilon\to+0}\left[\epsilon^{\frac{\ell(\ell+1)}{2}}\;e^{\frac{\ell(\ell+3)}{2}A(\epsilon)}\;\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\dots,z_{\ell+1}}(\underline{p}_{\ell+1})\right], \tag{3.9}$$

where $A(\epsilon)=-\frac{\pi^2}{6}\frac{1}{\epsilon}-\frac{1}{2}\ln\frac{\epsilon}{2\pi}$ *and* $x_i=x_{\ell+1,i}$, $i=1,\dots,\ell+1$.

Proof. We prove (3.9) by relating the recursive relation (1.5)

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\ldots,z_{\ell+1}}(\underline{p}_{\ell+1}) \;=\; \sum_{\underline{p}_\ell\in\mathcal{P}_{\ell+1,\ell}(\underline{p}_{\ell+1})} \Delta(\underline{p}_\ell)\; z_{\ell+1}^{\sum\limits_{i=1}^{\ell+1}p_{\ell+1,i}-\sum\limits_{j=1}^{\ell}p_{\ell,j}}\; Q_{\ell+1,\ell}(\underline{p}_{\ell+1},\underline{p}_\ell|q)\;\Psi^{\mathfrak{gl}_\ell}_{z_1,\ldots,z_\ell}(\underline{p}_\ell),$$

where

$$\begin{gathered} Q_{\ell+1,\ell}(\underline{p}_{\ell+1},\underline{p}_\ell|q) \;=\; \frac{1}{\prod\limits_{i=1}^{\ell}(p_{\ell+1,i}-p_{\ell,i})_q!\;(p_{\ell,i}-p_{\ell+1,i+1})_q!}\,, \\ \Delta(\underline{p}_\ell)=\prod_{i=1}^{\ell-1}(p_{\ell,i}-p_{\ell,i+1})_q!\,, \end{gathered} \tag{3.10}$$

with the recursive relation (3.3) for the classical Whittaker function.

Let us introduce the following parametrization of the elements of the Gelfand-Zetlin patterns $\underline{p}_\ell\in\mathcal{P}_{\ell+1,\,\ell}(\underline{p}_{\ell+1})$:

$$p_{\ell,\,k}\;=\;\epsilon^{-1}x_{\ell,\,k}\;+\;a_k m(\epsilon)\,,\qquad\qquad m(\epsilon)\;=\;-\big[\epsilon^{-1}\ln\epsilon\big], \tag{3.11}$$

where a_k are some constants. The Gelfand-Zetlin conditions on weights $\underline{p}_{\ell+1}$ reads as follows:

$$p_{\ell+1,\,k}\geq p_{\ell,\,k}\geq p_{\ell+1,\,k+1}\,,\qquad\qquad k=1,\ldots,\ell\,,$$

and they lead to

$$\epsilon^{-1}x_{\ell+1,\,k}+(\ell+2-2k)m(\epsilon)\;\geq\;\epsilon^{-1}x_{\ell,\,k}+a_k m(\epsilon)\geq\;\epsilon^{-1}x_{\ell+1,\,k+1}+(\ell-2k)m(\epsilon)\,. \tag{3.12}$$

The requirement that the limit $\epsilon\to+0$ preserves the conditions (3.12) implies the following restrictions on the parameters a_k:

$$\ell-2k+2\;>\;a_k\;>\;\ell-2k\,,\qquad\qquad k=1,\ldots,\ell\,. \tag{3.13}$$

Since $\underline{p}_\ell=(p_{\ell,1},\ldots,p_{\ell,\ell})\in\mathbb{Z}^\ell$ the only consistent choice in the limit $\epsilon\to+0$ is $a_k=\ell+1-2k$, $k=1,\ldots,\ell$. Although the variables $p_{\ell,\,k}$ are restricted to be in positive Weyl chamber i.e. $p_{\ell,k}\geq p_{\ell,k+1}$, in the limit $\epsilon\to+0$ the variables $x_{\ell,\,k}$ have no such restrictions. This follows from a simple observation that the limit $\epsilon\to+0$ the $\frac{a}{\epsilon}-b[\epsilon^{-1}\ln\epsilon]\to+\infty/-\infty$ depends only on the sign of non-zero coefficient b. Thus we have

$$p_{\ell,\,k}\;=\;\epsilon^{-1}x_{\ell,\,k}\;+\;(\ell+1-2k)m(\epsilon)\,. \tag{3.14}$$

Now using Lemma 3.1 it is easy to obtain the following limiting formulas:

$$\begin{aligned} \lim_{\epsilon\to+0}e^{2\ell A(\epsilon)}\mathcal{Q}_{\ell+1,\ell}(\underline{p}_{\ell+1},\underline{p}_\ell|q)\;&=\;\lim_{\epsilon\to+0}\frac{e^{2\ell A(\epsilon)}}{\prod\limits_{i=1}^{\ell}f_1(x_{\ell+1,\,i}-x_{\ell,\,i},\epsilon)\;f_1(x_{\ell,\,i}-x_{\ell+1,\,i+1},\epsilon)} \\ &=\;Q^{\mathfrak{gl}_{\ell+1}}_{\mathfrak{gl}_\ell}\big(\underline{x}_{\ell+1};\underline{x}_\ell;\,\lambda_{\ell+1}\big)\Big|_{\lambda_{\ell+1}=0}\,, \end{aligned} \tag{3.15}$$

$$\lim_{\epsilon\to+0}e^{(1-\ell)A(\epsilon)}\Delta(\underline{p}_\ell)\;=\;\lim_{\epsilon\to+0}e^{(1-\ell)A(\epsilon)}\prod_{i=1}^{\ell-1}f_2(x_{\ell,i}-x_{\ell,i+1},\epsilon)\;=\;1\,, \tag{3.16}$$

where $Q^{\mathfrak{gl}_{\ell+1}}_{\mathfrak{gl}_\ell}\big(\underline{x}_{\ell+1};\underline{x}_\ell;\,\lambda_{\ell+1}\big)$ is given by (3.4). This implies the following identity:

$$\lim_{\epsilon\to+0}\Big\{\epsilon^\ell\sum_{\underline{p}_\ell\in\mathcal{P}_{\ell+1,\ell}(\underline{p}_{\ell+1})} z_{\ell+1}^{\sum\limits_{i=1}^{\ell+1}p_{\ell+1,i}-\sum\limits_{j=1}^{\ell}p_{\ell,j}}\,\Big[e^{(\ell+1)A(\epsilon)}Q_{\ell+1,\,\ell}\big(\underline{p}_{\ell+1};\underline{p}_\ell|q\big)\,\Delta(\underline{p}_\ell)\Big]$$
$$\times\ \epsilon^{\frac{\ell(\ell-1)}{2}}e^{\frac{(\ell-1)(\ell+2)}{2}A(\epsilon)}\Psi^{\mathfrak{gl}_\ell}_{z_1,\ldots,z_\ell}(\underline{p}_\ell)\Big\}$$

$$\begin{aligned}=&\int\limits_{\mathbb{R}^\ell}d\underline{x}_\ell\ \exp\Big\{\imath\lambda_{\ell+1}\big(\sum_{i=1}^{\ell+1}x_{\ell+1,i}-\sum_{j=1}^{\ell}x_{\ell,j}\big)\Big\}\\ &\times\lim_{\epsilon\to+0}\Big[e^{(\ell+1)A(\epsilon)}Q_{\ell+1,\,\ell}\big(\underline{p}_{\ell+1}(\underline{x}_{\ell+1},\,\epsilon);\,\underline{p}_\ell(\underline{x}_\ell,\,\epsilon)\,\big|\,q(\epsilon)\big)\ \Delta\big(\underline{x}_\ell(\underline{x}_\ell,\,\epsilon)\big)\Big]\\ &\times\lim_{\epsilon\to+0}\Big[\epsilon^{\frac{\ell(\ell-1)}{2}}e^{\frac{(\ell-1)(\ell+2)}{2}A(\epsilon)}\Psi^{\mathfrak{gl}_\ell}_{z_1,\ldots,z_\ell}\big(\underline{p}_\ell(\underline{x},\,\epsilon)\big)\Big]\,.\end{aligned}\tag{3.17}$$

Thus we recover the recursive relations (3.3) for the Givental integrals directly leading to the integral representation (3.1) for the classical $\mathfrak{gl}_{\ell+1}$-Whittaker function. Using (3.17) iteratively over ℓ we obtain (3.9). $\Box$

Example 3.1 *For $\ell=1$ we have*

$$\Psi^{\mathfrak{gl}_2}_{z_1,z_2}(p_{2,1},p_{2,2})=\sum_{p_{2,2}\le p_{1,1}\le p_{2,1}}\frac{z_1^{p_{1,1}}\,z_2^{p_{2,1}+p_{2,2}-p_{1,1}}}{(p_{1,1}-p_{2,2})_q!(p_{2,1}-p_{1,1})_q!},\qquad p_{2,2}\le p_{2,1}\,,$$

$$\Psi_{z_1,z_2}(p_{2,1},p_{2,2})=0,\qquad p_{2,2}>p_{2,1}\,.$$

Using the parametrization

$$q=e^{-\epsilon},\qquad p_{21}=m(\epsilon)+x_{21}\epsilon^{-1}\qquad p_{22}=-m(\epsilon)+x_{21}\epsilon^{-1}\qquad z_i=e^{\imath\epsilon\lambda_i},\qquad i=1,2,$$

with $m(\epsilon)=-[\epsilon^{-1}\ln\,\epsilon]$ we obtain

$$\Psi^{\mathfrak{gl}_2}_{z_1,z_2}(p_{21},p_{22})=\sum_{x_{22}-\epsilon m(\epsilon)\le x_{11}\le x_{2,1}+\epsilon\, m(\epsilon)}\frac{e^{\imath\lambda_1x_{11}+\imath\lambda_2(x_{21}+x_{22}-x_{11})}}{((x_{11}-x_{22})/\epsilon+m(\epsilon))_q!\ ((x_{21}-x_{11})/\epsilon+m(\epsilon))_q!},$$

where we use the notations $p_{11}=x_{11}/\epsilon$. Taking into account

$$\frac{1}{(y/\epsilon+m(\epsilon))_q!}=e^{+\frac{\pi^2}{6}\frac{1}{\epsilon}+\frac{1}{2}\ln\frac{\epsilon}{2\pi}-e^{-y}+O(\epsilon)},$$

we obtain

$$\begin{aligned}\psi^{\mathfrak{gl}_2}_{\lambda_1,\lambda_2}(x_1,x_2)\ &=\lim_{\epsilon\to+0}\ \epsilon\,e^{-\frac{\pi^2}{3}\frac{1}{\epsilon}-\ln\frac{\epsilon}{2\pi}}\ \Psi^{\mathfrak{gl}_2}_{z_1,z_2}(p_{21},p_{22})\\ &=\int_{\mathbb{R}}dx_{11}e^{\imath\lambda_1x_{11}}e^{\imath\lambda_2(x_{21}+x_{22}-x_{11})}\,e^{-e^{x_{11}-x_{21}}-e^{x_{22}-x_{11}}}\,.\end{aligned}$$

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